FOCUS ISSUE: the value of "Why?"

3 Ways That Promote Student Reasoning

Encourage children and their future teachers to question whether different solution methods are mathematically valid and to justify their reasoning.

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Children are naturally curious and want to make sense of their world. To implement mathematical tasks that nurture children's desire to reason, it is valuable for teachers to have experienced for themselves comparable tasks and learning environments (Ball and Bass 2000). Within this article, I describe three strategies to facilitate reasoning in the context of lessons in number and operations. I have named these strategies *invent a way*, *imagine an alternative*, and *compare and contrast*. The strategies allow for students to examine problem-solving strategies that may be different from their own and, in some cases, may seem contradictory. As they question why one another's methods work, learners begin to build connections and increase flexibility in their own problem solving. Consider how these strategies generalize to promote reasoning by students of all ages.

Strategy 1: Invent a way

When mathematics students create their own ways of solving problems mentally, they are often encouraged by their peers to explain why their strategy makes sense. Because each student is constructing a method that may not be intuitive to others in the learning community, questions arise: How were you thinking about that? Why did that work?

Why does it work?

A brief example of a classroom discussion of the Wheatley and Reynolds's (2010) ten-frame task (see **fig. 1**, task 1a) demonstrates how early elementary students' different strategies promote opportunities to discuss the reasoning behind how someone saw the total so quickly. All student names are pseudonyms.

I'm only going to show the card for a few seconds, so get where you can see. Here it is! Kyle, how many dots did you see?

It was fast, but I think there's eight over there and then seven more over there, and that makes . . . [*counting quietly, nine, ten, eleven, twelve, thirteen, fourteen*] fifteen.

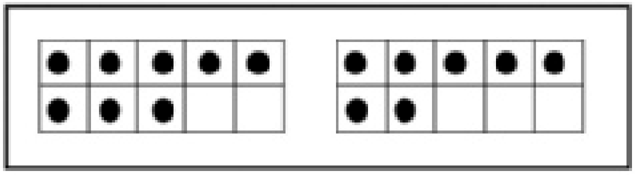
Did anyone else see fifteen? Maya? Tell us how you saw fifteen. Listen carefully, Kyle and everyone, because you want to see if Maya's explanation is the same as or different from the way you saw fifteen. Let's see if we can understand it.

Well, I think I saw two empty spaces on that side, so with my mind, I moved two dots over.

FIGURE 1

The task below encourages learners to use the invent-a-way strategy. Showing the ten-frame images from task 1a spurs children to mentally arrange the dots in different ways to make seeing the total easier. Preservice teachers are introduced to the strategy using task 1b to prompt explanations for alternate solutions to a problem.

Task 1a. The instructor shows students the ten-frame card for two or three seconds and then asks questions, one at a time: How many dots did you see? How did you see fifteen? Why did moving the dots in your mind make it easier to see how many were there?

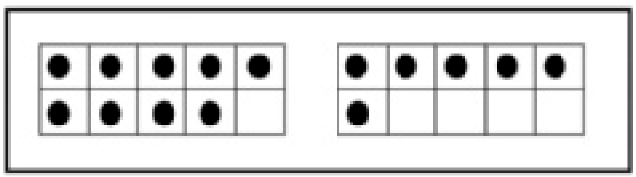


Task 1b. Suppose someone decides to solve $17\frac{3}{4}+3\frac{1}{2}$ by starting with $17\frac{1}{4}+4$. Explain the steps to solve the problem and reasons why the steps work.

FIGURE 2

Children (task 2a) and preservice teachers (task 2b) are encouraged to use the imagine-an-alternative-answer strategy.

Task 2a. The instructor briefly shows the ten-frame card to the students, anticipating at least two different answers.



Task 2b. Solve the subtraction problem $25\frac{1}{4}-4\frac{7}{8}$ by starting with 25 – 5 without using a standard algorithm. The instructor anticipates at least two different answers to arise.

Can you imagine what Maya was doing? Jayda, can you explain Maya's strategy?
She said she moved dots, but I'm not sure which dots. I didn't see it long enough.

Let's take another look at the card and see if we can figure this out. Jayda said Maya moved dots over. [*She shows* **fig. 1** *again.*] **And how much did that make over here, Kyle?**
Uhm . . . ten. They're all filled in.

Why did moving the dots help Maya see how many dots there were altogether?
[*Natalia*] It made a friendly number. Ten is easier.

So, when Maya moved two dots over, she made ten; but how many were left over here?
[*Jayda*] Five. Now I see it. Eight and seven is like ten and five. It's just like when we moved one dot over with 9 + 6, it made ten and left five, so it was fifteen dots altogether. I think you always move dots to the bigger number to make ten.

Having children invent their own methods of solving math problems offers several benefits. Students in this classroom are learning strategies from one another that will help them compute more fluently. At the same time, they are questioning why these ways work and under what conditions to use them. Note, too, that the instructor requested input from several students and asked them to build on one another's ideas.

Why is it valid?

As with children, adults also attempt to understand methods invented by their peers. However, adult learners often focus more on *how* their classmate got an answer or on other meanings of why they did. For example, the question of why you did it that way might be answered with any of the following explanations: Because that's the way I was taught; because it gave me the correct answer; because it was easier.

Instead, my colleagues and I try to help teacher candidates understand that more important questions are, Why is your solution valid? and What kind of argument could you provide to convince a skeptic? An invented solution method (see **fig. 1** task 1b) helped initiate the following discussion among teacher candidates.

When Jason remarked, "You're done! Both problems have the same answer, 21 1/4," Amal expressed her confusion. "But how can you tell that without working it out?"

Stephanie tried to explain: "Maybe it's like they took a half from 17 3/4 and gave it to 3 1/2 to make it 4 so it's easier.

Amal responded, "OK, so you could even move 1/4 over from 3 1/2 to make it 18 + 3 1/4. Both of those give easier problems, but will that always work?

At this point in the discussion, the instructor asked, "What kind of argument would convince

As prospective teachers share their solution strategies, they focus on important mathematical questions.

someone whether moving numbers around will always work?"

"I think it's OK," Stephanie responded, "'cause you're just finding the total amount in two piles. If I had 17 3/4 dollars and you had 3 1/2 dollars, we would have the same total even if I gave you a half dollar."

The explanations in this episode start by students describing *what* was done rather than *why* the method is valid. This happens as learners examine unfamiliar invented strategies. As the instructor rephrases the task to convince someone that the method will always work, the future teachers refer to a meaning of addition (joining) to justify the expression and a reason why it is equivalent.

Strategy 2: Imagine an alternative

Errors arise naturally in learning situations. With the imagine-an-alternative-answer strategy (see **fig. 2** task 2a), errors are used as springboards to discuss important underlying concepts. Editing and revising work is also highlighted as part of the work of doing mathematics.

Errors as springboards

The instructor begins the dialogue:

This card is a little tricky. I know some of you got sixteen and some got fifteen. This time, we're not going to check it right away, but we're going to think about what you think the person saw that would make them say sixteen or fifteen. Jeremiah, why do you think someone might have seen sixteen dots?

Well, maybe if that ten-frame was full on the left, and I think there were six more dots on the right, because I saw five and one, so 10 + 6 = 16.

What kind of thinking might have made someone say there were only fifteen dots?

[*Callie*] I'm pretty sure there was an empty box on the left frame, so you could move the extra dot over and then you would have a full ten-frame, but you'd only have five dots left, not six, so that would be fifteen altogether.

So, Jeremiah gave us a reason that someone might have seen sixteen dots, and Callie helped us understand how someone might have seen fifteen dots; and it all had to do with whether the first frame was full or not.

In this classroom episode, the focus is not on producing *the* answer, but instead on the reasoning that goes into an answer. The instructor remains neutral on the correctness of either answer until the students have had a chance to grapple with why each might be plausible. Within this discussion, the underlying mathematical idea of making ten is reinforced. Note that this teaching approach honors students' thinking, and asks students to try to understand their peers' thinking; at the same time, the approach helps students focus on important concepts (Sherin, Louis, and Mendez 2000).

Bringing reasoning to the fore

Erroneous solutions emerge in classrooms of adult learners just as in elementary school classrooms, but because of the stigma associated with errors, adults often decline to share their solutions. Since errors create valuable opportunities to summon reasoning and to refine and revise thinking, the instructor can introduce them as plausible alternatives (see **fig. 2**, task 2b). Bringing reasoning to the fore

→ reflect and discuss

"3 Ways That Promote Student Reasoning"

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms, and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions, related to "3 Ways That Promote Student Reasoning" by Margaret Rathouz, are suggested prompts to aid you in reflecting on the article, discussing it with your colleagues, and using the author's ideas to benefit your own classroom practice.

Rathouz presents three strategies to help promote classroom discussions around why different, apparently conflicting, solutions work. Each strategy was demonstrated in two different kinds of classrooms; all the examples were taken from the Numbers and Operations strand.

A. Consider how these same strategies could play out with your particular students in a variety of mathematical content areas, including geometry, measurement, probability, data analysis, or algebra. Below are a few prompts to help you appreciate how these strategies might appear in other settings.

Invent-a-way strategy

1. What method could you use to find the area of a trapezoid? Why does your way work?
2. Sort these shapes in a way that makes sense to you. Try to figure out the properties used by others to sort the same shapes differently.

Imagine-an-alternative-answer strategy

1. Is fraction division with a remainder a plausible strategy? Why, or why not?
2. What does this organizational scheme say about the definition being used for *trapezoid*?
3. One student found the formula for the volume of a sphere to be $V = \frac{2}{3}\pi r^2 h$, whereas another student found the formula to be $V = \frac{4}{3}\pi r^3$. Could they both be correct? How?

Compare-and-contrast strategy

1. Why does the decimal point sometimes "move" for multiplication but not for division?
2. How can the mean of a data set be higher than the median? Under what conditions does that happen?
3. Under what conditions do you add the probabilities of two events happening? Under what conditions do you multiply the probabilities? Why?

B. What other strategies do you and your colleagues use regularly to stimulate mathematically rich discussions?

C. Reflect on and discuss several of the benefits of these and other strategies that might reveal errors and misconceptions.

Tell us how you used Reflect and Discuss as part of your professional development. Submit letters to *Teaching Children Mathematics* at **tcm@nctm.org**. Include Readers Exchange in the subject line. Find more information and submission guidelines at **tcm.msubmit.net**.

helps learners begin to appreciate that making sense of processes leads to greater accuracy in problem solving.

So, I've been walking around as you were working, and I've seen two different answers, 20 1/8 and 20 3/8, for the problem. What I want you to do now is figure out how might somebody have gotten a different answer from you? What was their thinking to get these two different answers?
[*Amber*] I tried to think how someone got 20 1/8. I think they changed the numbers to whole numbers and did 25 – 5 = 20. Then, since they took 1/4 off of 25 1/4, they needed to add it back on to 20 to get 20 1/4. And since they added 1/8 to get 5, they had to take it away: 20 1/4 – 1/8 = 20 1/8.

Jessica, you look like you would like to advocate for the other answer.
I started out the same way, but in the end, I added 1/8 to 20 1/4 to get 20 3/8.

Sometimes this happens—that we have two different answers—and Amber helped us to see why someone might have gotten 20 1/8, but Jessica did something different and got 20 3/8. So, now it's up to all of us to reason about what this problem 25 1/4 – 4 7/8 means so we can figure out which answer makes sense. Why don't you take a few minutes with your group to sort out this dilemma?
[*The instructor provides time for small-group discussion.*]
[*Brittany*] So, we were thinking about 25 1/4 – 4 7/8 as, like, I have about 25 gallons of gas in my tank, and then I drive somewhere, and that uses up around 5 gallons of gas; so how much gas do I have left? You know it's going to be about 20 gallons.

Is everyone following how Brittany's group was thinking about it? Who can continue with this line of reasoning to help decide where to go from here? Megan?
Well, since we actually started with more than 25 gallons, we should add the 1/4 gallon.
[*Cara*] But we didn't really use all 5 gallons—only 4 7/8 gallons—so maybe we should make up for that by adding 1/8 to 20 1/4 gallons, instead of subtracting 1/8.

To facilitate rich discussion, teaching requires mathematical knowlege.

[*Amber*] When I look at the equation, it still seems like you should subtract 1/8. But I guess when you think about the gas situation, I can kind of see why they needed to add 1/8. We actually took too much away!

Building on the ideas of others requires participants to follow explanations and answers that are different from their own. The instructor plays a role in reminding students of tools they might use to make sense of fractions and subtraction (e.g., contexts, diagrams, meanings of the numbers and operations) so they can reason why both answers seem plausible, but only one is correct. The other students play the critical role of generating an argument to convince their peers.

Strategy 3: Compare and contrast

A comparison situation affords a third opportunity to invoke mathematical reasoning to resolve what might appear at first to be a paradox. In the next two cases, students compare what works for creating equivalent problems with two different operations.

Why here but not there?

Through reasoning about the roles of numbers in a subtraction problem, the children begin to compare why certain processes are valid for addition and others for subtraction.

We've been working on using friendly numbers for a while to help us figure out problems like 15 + 9. [*See* fig. 3, *task 3a.*] Sara, can you remind us how that strategy works?
I usually think about making 10 or 20 or 30. For 15 + 9, I would move 1 from the 15 to the 9 to make 10. Then, I have 14 + 10, and that's 24.

So, what I want you to figure out now is, does that same thing work for subtraction? Can we change numbers to friendly numbers to make a problem like 15 – 9 easier? What do you think, Aiden?
I changed 15 – 9 to 14 – 10, and that's easy to subtract; it's 4.
[*Kyle*] But that's not right, 'cause if you have 15 and you take away 9, it's going to be more.

Jeremiah, it looks like you have an idea how to build on what Kyle just said.
I was thinking about those ten-frames. Fifteen dots would be a whole ten-frame and five more dots. And since we're taking away only nine . . . .
[*Aiden*] Can I revise my answer? What Jeremiah said made me think that I would take off nine

dots, but there would still be one dot left in that frame and five more in the other one, so six dots.

[*Natalia*] I have an idea about subtraction. You know how we always took a dot off one ten-frame to fill in an empty space when we were adding nine? Maybe when you take away nine, you have an extra dot.
That is an interesting conjecture. We should all think more about it.

The culture of the classroom encourages these students to build on their own prior knowledge and others' mathematical ideas. Thames and Ball (2010) have noted the mathematical knowledge needed by the teacher to facilitate such discussions. Such knowledge can be initiated in preservice teachers' own discussions of a related problem, as seen below.

Rules without reasons

In the dialogue below, the future teachers grapple with a problem (see **fig. 3**, task 3b) similar to the one the children discussed (see **fig. 3**, task 3a). Comparing addition and subtraction causes the adult learners to revisit what each operation means and provide a convincing argument, based on those meanings, as to why one method works to create an equivalent addition problem and a different one works to create an equivalent subtraction problem.

[*After discussing task 1b*] **It seems like some of you are suggesting moving part of one number to the other number to make an easier problem with the same answer. Would that same thing work for a subtraction problem? In other words, is 17 3/4 – 3 1/2 equivalent to 17 1/4 – 4?** [*Giving students a few minutes to work out their answers*] **What did your groups figure out?**
[*Jennifer*] Our group found out that it doesn't really work for the subtraction problem.
[*David*] Yeah, we think that, with subtraction, you have to do the same thing to both sides.

What do you think of David's idea? What does he mean by *both sides*?
[*Angela*] Like, if you add 1/2 to 3 1/2, then you have to add 1/2 to 17 3/4, too. We think that works for subtraction, but with addition, you can move numbers around.
[*Jennifer*] We were thinking that, too, but those just sound like rules. Why is it one rule with this problem and another rule with the other problem?

Why is it that one strategy works for making an equivalent problem with addition and another strategy works for subtraction?
[*Melanie*] Maybe for 17 3/4 – 3 1/2, we could see how far it is from 3 1/2 to 17 3/4 on a number line? And then, you can see it's the same as between 4 and 18 1/4? Can I draw it? [*Melanie explains her diagram* (see **fig. 4**) *by showing with her hands how far 3 1/2 and 17 3/4 are from each other. Then she shifts over to 4 and 18 1/4, keeping her hands the same distance apart.*]
[*Jennifer*] They are telling you to do two different things. It's like with addition, you can just put the two numbers together in any order 'cause you're finding the total amount. But Melanie's picture is finding the distance between the two numbers because subtraction is finding the difference.

These prospective elementary teachers are beginning to appreciate that rules without reasons are not satisfying. To help their own future students understand why the mathematics is

FIGURE 3

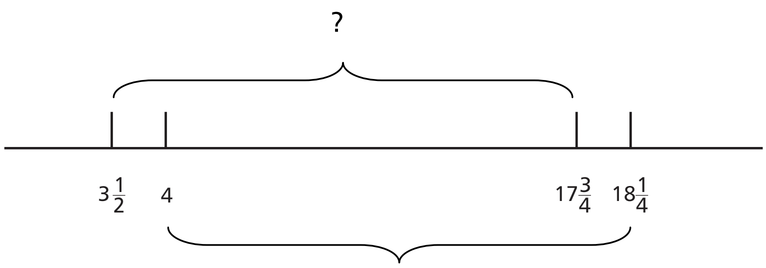
Tasks use the compare-and-contrast strategy to promote children (3a) and adult (task 3b) learners to explain why.

Task 3a. The sum of 15 + 9 is equivalent to 14 + 10. Investigate whether using 10 instead of 9 can be used to create an easier problem equivalent to 15 – 9.

Task 3b. Explore whether generating equivalent subtraction problems can use the same strategy as with equivalent addition problems.

FIGURE 4

The diagram shows the equivalence of $17\frac{3}{4} - 3\frac{1}{2}$ and $18\frac{1}{4} - 4$.



Children will enthusiastically participate in mathematical discussions in which their thinking is valued.

valid, they themselves first have to make sense of what it means.

Final thoughts

The conscious enactment of instructional tasks in ways that encourage conflicting or ambiguous ideas to emerge creates opportunities for learners—children and their future teachers alike—to understand the importance of justifying why their thinking is valid (Grant, Lo, and Flowers 2007). The invent-a-way strategy relies on diversity in the student's invented solutions to motivate peers to request clarification. The imagine-an-alternative strategy introduces common correct and incorrect solutions that are often latent in the group. This focuses the discussion on the reasoning behind an answer rather than its correctness. With the compare-and-contrast strategy, the mathematical ideas in two scenarios appear to be contradictory, and that contradiction prompts students to question why things work the way they do. When students are asked to interpret, build on, or revise others' mathematical ideas that may be in conflict with their own, they strengthen their own abilities to generate convincing explanations for why the mathematics works.

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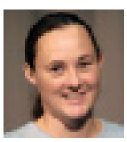

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